Is the cosmological singularity really unavoidable in general relativity?

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The initial singularity problem in standard general relativity is treated on the light of a viewpoint asserting that this formulation of Einstein's theory and its conformal formulations are physically equivalent. We show that flat Friedmann-Robertson-Walker universes and open dust-filled and radiation-filled universes are singularity free when described in terms of the formulation of general relativity conformal to the canonical one.

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One of the most undesirable features of general relativity is the ocurrence of spacetime singularities where the laws of physics breakdown. The famous Hawking-Penrose singularity theorems predict that, if ordinary matter obeys some reasonable energy conditions, then the ocurrence of spacetime singularities in general relativity is inevitable [1]. In particular, the initial singularity is a problem in standard general relativity given by the action:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-\hat{g}}(\hat{R} + 16\pi\hat{L}_{matter}) \tag{1}$$

where $\hat{R}$ is the Ricci scalar of the metric $\hat{g}_{ab}$ and $\hat{L}_{matter}$ is the Lagrangian for ordinary matter.

Recent developments of string theory suggest that the initial singularity can be resolved by including new degrees of freedom like p-brane [2]. However, the low-energy limit of string theory is usually linked with Brans-Dicke gravity so, the new degree of freedom is treated, precisely, in the frame of this theory. In particular, in reference [3], it has been argued that the gas of solitonic p-brane treated as a perfect-fluid-type matter in Brans-Dicke theory can resolve the initial singularity without any explicit solution.

In this letter we shall treat the initial singularity problem in the light of a viewpoint, first presented in reference [4] for general relativity in empty space, according to which the usual formulation of Einstein's theory given by the action (1) and its conformal formulation given by:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}[\phi R + \frac{3}{2}\phi^{-1}(\nabla\phi)^2 + 16\pi L_{matter}] \tag{2}$$

are physically equivalent. In eq.(2) R is the Ricci scalar given in terms of the metric g_{ab} that is conformal to $\hat{g}_{ab}$:

$$\hat{g}_{ab} = \phi g_{ab} \tag{3}$$

ϕ is some scalar function given on the manifold and $L_{matter} = \phi^2\hat{L}_{matter}$ is the Lagrangian for ordinary matter nonminimally coupled to the scalar ϕ. In what follows we shall call the formulation of general relativity due to (1) as Einstein frame general relativity while, its conformal formulation due to (2), we call as Jordan frame general relativity, a terminology usual in scalar-tensor gravity.

The physical equivalence of Einstein frame general relativity and Jordan frame one is supported by the fact that physical experiment is not sensitive to the transformation (3), that can be interpreted as a particular units transformation [5]. Actually, measurements of dimensional quantities represent ratios with respect to standard units so the measurables of the theory are always dimensionless numbers[1] and, then, are unchanged under the units transformation (3). In particular, the dimensionless gravitational coupling constant $Gm^2 (\hbar = c = 1)$, where m is the rest mass of some particle and G is the dimensional gravitational constant, is kept unchanged under the transformation (3) [5]. Then, the fact that Gm^2 is a constant in general relativity, is a conformal invariant feature of this theory. In the formulation due to (1) both, G and m, are constant over the manifold, while in the conformal formulation due to (2), both G and m are variable in spacetime: $G \sim \phi^{-1}$ and $m \sim \sqrt{\phi}$.

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[1]For a readable discussion on the dimensionless nature of measurements see section II of reference [6]

Our viewpoint leads that the geometrical interpretation of a given cosmology through general relativity is not just one unique picture of it, but a whole equivalence class of all conformally related cosmologies [7].

From (1) the following equation can be derived:

$$\hat{G}_{ab} = 8\pi \hat{T}_{ab} \tag{4}$$

where $\hat{T}_{ab} = \frac{2}{\sqrt{-\hat{g}}}\frac{\partial}{\partial \hat{g}^{ab}}(\sqrt{-\hat{g}}\hat{L}_{matter})$ is the stress-energy tensor for ordinary matter fulfilling the conservation equation:

$$\hat{\nabla}_n \hat{T}^{na} = 0 \tag{5}$$

The field equations derivable from the action (2) are:

$$G_{ab} = 8\pi\phi^{-1}T_{ab} - \frac{3}{2}\phi^{-2}[\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}(\nabla\phi)^2] + \phi^{-1}[\nabla_a\nabla_b\phi - g_{ab}\Box\phi] \tag{6}$$

and

$$\Box\phi - \frac{1}{2}\phi^{-1}(\nabla\phi)^2 - \frac{1}{3}R\phi = \frac{8\pi}{3}T \tag{7}$$

where $G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R$.

Once the components of the metric $\hat{g}_{ab}$ are known, we can find the components of the conformal metric g_{ab} by using the relationship (3), i.e., without solving equation (6). For this purppose we should find the functional form of the scalar function ϕ with the help of eq.(7). Considerable simplification of eq.(7) can be achieved if we look for particular solutions with the curvature scalar given by:$R = -8\pi T - \frac{3}{2}\phi^{-2}(\nabla\phi)^2$. In this case the scalar function ϕ can be found as solution to the wave equation: $\Box\phi = 0$. Under (3) this equation is mapped into:

$$\hat{\Box}\hat{\phi} = 0 \tag{8}$$

with $\hat{\phi} = \ln\phi$. This last wave equation can be treated by considering the metric with hat $\hat{g}_{ab}$ that is solution to the Einstein frame field equation (4).

We shall interested here in Friedmann-Robertson-Walker (FRW) universes with the line element($d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$):

$$d\hat{s}^2 = -dt^2 + \frac{\hat{a}(t)^2}{1-kr^2}(dr^2 + r^2 d\Omega^2) \tag{9}$$

where $\hat{a}(t)$ is the Einstein frame scale factor and $k = +1$ for closed,$k = 0$ for flat and $k = -1$ for open universes. Suppose the universe is filled with a perfect-fluid-type matter with the barotropic equation of state: $\hat{p} = (\gamma - 1)\hat{\rho}$, $0 < \gamma < 2$. In this case the time component of the conservation equation (5) can be integrated to give: $\hat{\rho} = \frac{(C_2)^2}{\hat{a}^{3\gamma}}$, where $(C_2)^2$ is some integration constant. The (0,0) component of equation (4) can then be written as:

$$(\frac{\dot{\hat{a}}}{\hat{a}})^2 + \frac{k}{\hat{a}^2} = \frac{8\pi}{3}\frac{(C_2)^2}{\hat{a}^{3\gamma}} \tag{10}$$

the overdot means derivative with respect to the Einstein frame proper time t. Equation (8), for his part, can be integrated to give:

$$\dot{\hat{\phi}} = \pm\frac{C_1}{\hat{a}^3} \tag{11}$$

where C_1 is another integration constant.

Our goal here is to compare spacetimes given by the line element (9) and its conformal:

$$ds^2 = -dt^2 + \frac{a(t)^2}{1-kr^2}(dr^2 + r^2 d\Omega^2) \tag{12}$$

where the Jordan frame proper time τ and its conformal Einstein frame proper time t are related through:

$$\tau - \tau_0 = \int dt \exp(-\frac{1}{2}\hat{\phi}) \tag{13}$$

while the Jordan frame scale factor a is related with the Einstein frame one through:

$$a = \hat{a} \exp(-\frac{1}{2}\hat{\phi}) \tag{14}$$

The procedure to be followed here is, first to solve equation (10) in order to find the time dependence of $\hat{a}$ and, then we integrate equation (11) to obtain $\hat{\phi}$. Finally, with the help of equations (14) and (13) we find the functional form of the Jordan frame scale factor and the functional relationship between τ and t respectively.

We shall, first, investigate flat universe with $k = 0$ and then open universes with $k = -1$. Finally, closed universes with $k = +1$ will be treated in outline.

For flat FRW universes the solution to eq.(10) is found to be:

$$\hat{a}(t) = A^{\frac{2}{3\gamma}} t^{\frac{2}{3\gamma}} \tag{15}$$

where the constant factor $A = \sqrt{6\pi}\gamma C_2$ while, after integrating eq.(11) we obtain:

$$\hat{\phi}^{\pm}(t) = \hat{\phi}_0 \mp \frac{C_1\gamma}{A^{\frac{2}{\gamma}}} \frac{t^{1-\frac{2}{\gamma}}}{2-\gamma} \tag{16}$$

Consequently, the Jordan frame scale factor is given by the following expression:

$$a^{\pm}(t) = \frac{A^{\frac{2}{3\gamma}}}{\sqrt{\phi_0}} t^{\frac{2}{3\gamma}} \exp[\pm \frac{C_1\gamma}{2A(2-\gamma)} t^{1-\frac{2}{\gamma}}] \tag{17}$$

The proper τ and its conformal are related through:

$$(\tau - \tau_0)^{\pm} = \frac{1}{\sqrt{\phi_0}} \int dt \exp[\pm \frac{C_1\gamma}{2A(2-\gamma)} t^{1-\frac{2}{\gamma}}] \tag{18}$$

In the Jordan frame our solution has two branches ('+' branch and '-' branch), given by the choice of the '+' or '-' signs in (17) and (18). The evolution of the '-' branch Jordan frame FRW universe is basically the same as that of its conformal Einstein frame univerese: it evolves from a cosmological singularity at $\tau = \tau_0$ ($t = 0$) into an infinite size universe at the infinite future $\tau = +\infty$ ($t = +\infty$). It is the usual picture in canonical general relativity where the cosmological singularity is unavoidable. When we work in the '+' branch of the Jordan frame solution, for his part, the flat FRW perfect-fluid-filled universe evolves from an infinite size at the infinite past $\tau = -\infty$ ($t = 0$) into an infinite size at the infinite future $\tau = +\infty$ ($t = +\infty$) through a bounce at $t^* = [\frac{3}{4}\frac{\gamma C_1}{A^{2\gamma}}]^{\frac{\gamma}{2-\gamma}}$ where it reaches its minimum size: $a^* = \frac{1}{\sqrt{\phi_0}}[\sqrt{\frac{3}{32\pi}}\frac{C_1}{C_2}e]^{\frac{2}{3(2-\gamma)}}$, i.e., the '+' branch flat FRW universe with the line element (12) is free of the cosmological singularity [8].

Now we shall study open FRW universe ($k = -1$). In this case analytic expressions for the scalar function $\hat{\phi}$ and, correspondingly, for the Jordan frame scale factor can be reached only for particular values of the parameter γ. We shall study only two generic cases: $\gamma = 1$ (dust-filled universe) and $\gamma = \frac{4}{3}$ (radiation filled universe).

For a dust-filled universe we have:

$$\hat{a}(\eta) = \eta^2 - B^2 \tag{19}$$

where the constant $B = \sqrt{\frac{8\pi}{3}}C_2$ and we have introduced the time coordinate η: $d\eta = \hat{a}^{-\frac{1}{2}}dt$. For $\hat{\phi}$ we obtain:

$$\hat{\phi}^{\pm}(\eta) = \hat{\phi}_0 \pm \frac{9C_1\eta}{B^4}\frac{(\eta^2 - 9B^2)}{(\eta^2 - B^2)^{\frac{3}{2}}} \tag{20}$$

so, the Jordan frame scale factor is given by:

$$a^{\pm}(\eta) = \frac{1}{\sqrt{\phi_0}}(\eta^2 - B^2)\exp[\mp\frac{9C_1\eta}{2B^4}\frac{(\eta^2 - 9B^2)}{(\eta^2 - B^2)^{\frac{3}{2}}} \tag{21}$$

while (13) gives:

$$(\tau - \tau_0)^{\pm} = \frac{1}{\sqrt{\phi_0}}\int d\eta\sqrt{\eta^2 - B^2}\exp[\mp\frac{9C_1\eta}{2B^4}\frac{(\eta^2 - 9B^2)}{(\eta^2 - B^2)^{\frac{3}{2}}} \tag{22}$$

Once again the '-' branch of our solution for a Jordan frame open universe and its conformal Einstein frame universe evolve in a similar fashion: they evolve from a cosmological singularity at $\tau = \tau_0$ and $\eta = B$ respectively, into an infinite size universe at the infinite future $\tau = \eta = +\infty$. When we work in the '+' branch of the Jordan frame solution, for his part, the FRW universe evolves from an infinite size at the infinite past $\tau = -\infty$ ($\eta = B$) into an infinite size at the infinite future $\tau = +\infty$ ($\eta = +\infty$) through a minimum size at some intermediate time. A similar situation we found when we study an open FRW radiation-filled universe given, in the Einstein frame, by the line element (9) with:

$$\hat{a}(t) = \sqrt{t^2 - B^2} \tag{23}$$

In this case:

$$\hat{\phi}^{\pm}(t) = \hat{\phi}_0 \mp \frac{C_1 t}{B^2\sqrt{t^2 - B^2}} \tag{24}$$

so, the Jordan frame scale factor is found to be:

$$a^{\pm}(t) = \frac{\sqrt{t^2 - B^2}}{\sqrt{\phi_0}}\exp\pm\frac{C_1 t}{2B^2\sqrt{t^2 - B^2}} \tag{25}$$

while the relationship (13) can be written as:

$$(\tau - \tau_0)^{\pm} = \frac{1}{\sqrt{\phi_0}}\int dt\exp\pm\frac{C_1 t}{2B^2\sqrt{t^2 - B^2}} \tag{26}$$

The analysis of this situation (radiation-filled universe with $k = -1$) shows that, as in the former cases, the cosmological singularity inherent to the Einstein frame FRW universe is removed in the '+' branch of the Jordan frame solution.

We get that flat FRW universes and open FRW dust-filled and radiation-filled universes are free of the cosmological singularity when treated with the help of the Jordan frame formulation of general relativity given by the action (2). In the light of the viewpoint developed in this letter both, Einstein frame FRW universes with $k = 0$ and $k = -1$ showing a cosmological singularity and their Jordan frame, singularity free, counterparts are experimentally undistinguishable, i.e., physically equivalent. When one chooses the Einstein frame picture for the description of a given cosmology, its geometrical interpretation should be given in terms of Riemann geometry with constant units of measure. If, in place of this, one chooses the Jordan frame formulation of general relativity for the interpretation of the given cosmology then, the corresponding geometrical interpretation should be formulated in terms of a more general geometry with the units of measure varying length in spacetime. This leads, in particular, that both the gravitational constant G and the rest mass of any particle are variable over the FRW spacetime in this last formulation of general relativity. The

advantage of the Jordan frame formulation over the Einstein frame one is clear when we approach the cosmological singularity inherent to this last picture.

For closed universes the Jordan frame picture shows a cosmological singularity (in the past or in the future) for both branches of the solution so, in this case, the compact Einstein frame universe with $k = +1$ is mapped into an open one where the cosmological singularity can not be removed by the conformal transformation (3). We can interpret this result (in the light of the viewpoint developed in the present letter) as casting doubts on the validity of closed FRW spacetimes as physically realizable options for the description of our real universe. The details of this work will be given elsewhere.

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